We prove (2.1.46). In fact we prove (2.1.46) for F in $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, which covers both cases $p = 2$ and $p = \infty$, that we need.

Let N_f be the Lebesgue set of f. Then for fixed $x \in N_f$ and given $\varepsilon > 0$ there is a $\delta_0 > 0$ such that

$$0 < r \le \delta_0 \implies \frac{1}{v_n r^n} \int_{|y|<r} |F(x-y) - F(x)|\, dy < \varepsilon.$$

Write

$$(P_t * F)(x) - F(x) = (P_t * F)(x) - \Big(\int_{\mathbf{R}^n} P_t(y)\, dy \Big) f(x) = \int_{\mathbf{R}^n} P_t(y) \big(F(x-y) - F(x) \big)\, dy$$

and we bound this by

$$\int_{|y|<\delta_0} |F(x-y) - F(x)|\, P_t(y)\, dy + \int_{|y|\ge\delta_0} |F(x-y) - F(x)|\, P_t(y)\, dy.$$

We begin by estimating

$$\begin{aligned} \int_{|y|\ge\delta_0} |F(x-y) - F(x)|\, P_t(y)\, dy &\le \big\|F\big\|_{L^p} \big\|P_t\big\|_{L^{p'}(|y|\ge\delta_0)} + |F(x)|\, \big\|P_t\big\|_{L^1(|y|\ge\delta_0)} \\ &\le C_n A \left[\big\|F\big\|_{L^p} \frac{t}{\delta_0} \Big(\frac{1}{\delta_0}\Big)^{\frac{n}{p}} + |F(x)| \frac{t}{\delta_0} \right] \\ &< \varepsilon, \end{aligned}$$

provided $0 < t < \delta_1$ for some δ_1 (depending on F, x, and δ_0). Here we used the estimate

$$P_t(y) \le \frac{1}{t^n} \frac{A}{(1 + |x|/t)^{n+1}}$$

where A depends only on n.

Also, setting

$$G(r) = r^{n-1} \int_{\mathbf{S}^{n-1}} |f(x - r\theta) - f(x)|\, d\theta,$$

so that

$$\int_0^r G(\rho)\, d\rho = \int_0^r \rho^{n-1} \int_{\mathbf{S}^{n-1}} |F(x-\rho\theta) - F(x)|\, d\theta\, d\rho = \int_{|y|<r} |F(x-y) - F(x)|\, dy,$$

we write (using polar coordinates and an integration by parts)

$$\begin{aligned}
\int_{|y|<\delta_0} & |F(x-y)-F(x)|\,|P_t(y)|\,dy \\
&\le \int_{|y|<\delta_0} |F(x-y)-F(x)| \frac{1}{t^n}\frac{A}{(1+|y|/t)^{n+1}}\,dy \\
&= \int_0^{\delta_0} \frac{d}{dr}\Big[\int_0^r G(\rho)\,d\rho\Big]\frac{1}{t^n}\frac{A}{(1+r/t)^{n+1}}\,dr \\
&= \left(\int_0^{\delta_0} G(\rho)\,d\rho\right)\frac{1}{t^n}\frac{A}{(1+\delta_0/t)^{n+1}} - \int_0^{\delta_0}\left(\int_0^r G(\rho)\,d\rho\right)\frac{1}{t^{n+1}}\frac{-(n+1)A}{(1+r/t)^{n+1}}\,dr \\
&< \varepsilon\, v_n \,\frac{\delta_0^n}{t^n}\frac{A}{(1+\delta_0/t)^{n+1}} + \varepsilon\,(n+1)\,v_n \int_0^{\delta_0}\frac{1}{t^{n+1}}\frac{Ar^n}{(1+r/t)^{n+2}}\,dr \\
&< \varepsilon\, v_n\, A + \varepsilon\,(n+1)\,v_n\, A\int_0^\infty \frac{dr}{(1+r)^{n+2}} \\
&= C_n'\,\varepsilon.
\end{aligned}$$

Then for $0<t<\min(\delta_0,\delta_1)$ we deduce that $|(P_t*F)(x)-F(x)|<(C_n+C_n')\varepsilon$.